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Solving forward Lorentz–Dirac-like equations

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Abstract. It is shown that successive approximations can be used to implement a numerical method to integrate forward the Lorentz–Dirac equation, as well as other equations with the same singular structure. The code automatically selects the physical solution and avoids the so-called 'runaway solutions'. The method's convergence is analytically discussed in a particular but illustrative case. The convergence is also numerically studied in a capture motion and a chaotic scattering process.

1. Introduction

When the radiation reaction is taken into account the classical equation of motion of a point charge is the Lorentz–Dirac equation:

$$\ddot{x}^\mu = f^\mu + \tau_0 \left(\dddot{x}^\mu - \frac{\ddot{x}^\nu \ddot{x}_\nu}{c^2} \dot{x}^\mu \right) \tag{1}$$

where a dot indicates derivative with respect to the proper time, τ, $f^\mu = f^\mu(\tau, x, \dot{x})$ is the external force per unit mass and $\dot{x}^\mu \dot{x}_\mu = -c^2$. If the external force is the Lorentz force exerted by an electromagnetic field $F^{\mu\nu}$, we have

$$f^\mu = \frac{e}{mc} F^{\mu\nu} \dot{x}_\nu. \tag{2}$$

The parameter

$$\tau_0 \equiv \frac{2e^2}{3mc^3} \tag{3}$$

is, up to a numerical factor, the time that the light needs to travel across a classical electron radius. Since this defines a very small scale, in many cases one has to use the quantum theory instead of equation (1), but the latter is a good approximation in some astrophysical contexts [1].

It is well known that the Lorentz–Dirac equation has some puzzling properties [2, 3], but we will discuss here only two of the more striking ones: the equation order and the so-called 'runaway solutions'. Since it is a third-order equation, the ordinary Newtonian initial conditions (position and velocity) are not enough to select a solution, one also has to give the initial acceleration. But the Lorentz–Dirac equation is singular, in the limit $\tau_0 \to 0$ we recover a second-order equation. On the other hand, most of the solutions do not describe any physical motion and must be discarded because they are of the runaway type, i.e. the acceleration grows exponentially in time.

Except in some simple cases [2–4] one is not able to find explicit solutions for the Lorentz–Dirac equation and some approximation scheme has to be used. One may use

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some singular perturbation technique to solve a particular problem, but we are interested here in a general-purpose numerical algorithm to solve the equation for different external forces with ease and high accuracy. It is obvious that the two properties discussed above represent a serious obstacle to numerical integration. Even if one has devised a method to select one of the exceptional initial conditions that lead to a physical solution, the numerical solution will shift to a runaway one as a consequence of the unavoidable numerical errors [4–6].

The standard solution to this problem is to impose an asymptotic condition in the future infinity, usually $\lim_{\tau\to\infty} a^\mu = 0$, and to integrate backwards in time. The asymptotic condition effectively reduces the initial conditions to position and velocity and integrating backward the runaway modes are automatically damped. Recently, Comay [7] has used a more complete approximation method to describe the asymptotic motion in the remote future. This kind of method is very useful when it can be applied, but has severe limitations.

First, it can only be directly applied to scattering processes, in which the asymptotic motion is known. If one wants to study a capture process, the motion in the remote future from which to start the backward integration is not known *a priori*. In principle, one could use further approximations to describe the asymptotic future in each particular case. In fact, this has been done by Sawada [8] for a modified Coulombian potential, but an *ad hoc* charge distribution had to be assumed for the potential source.

Even in scattering cases, imposing initial conditions in the future infinity is rather contrived. Usually what is known is the initial motion, not the final one. For instance, let us assume that we want to study the scattering of a test charge by three (or more) stationary charged centres. The incoming direction and velocity in the infinite past are given and one has to compute the scattering angle for different impact parameters. As we will see, this kind of scattering is chaotic [9] and the scattering function has a Cantor set of discontinuities. Since the test particle spends a very long time in the scattering region for impact parameters near the discontinuity values, one expects the radiated energy to be important there and the velocity in the remote future will be very different from the incoming one. One could use some shooting method to estimate the correct final velocity and scattering angle for each impact parameter, but the computing power necessary to compute the scattering function will be very high due to the sensitive dependence on the initial conditions.

The aim of this paper is to show that there is a better alternative. By using a reasonable physical assumption and a method of successive approximations it is possible to devise a very general numerical method which allows us to integrate forward the Lorentz–Dirac equation, as well as other equations with the same singular structure. The physical assumptions and the approximation method are not new, but the convergence of the method has not been discussed and its analytical implementation is very cumbersome and has not been used to perform actual calculations. The contribution of this work is twofold: we discuss the convergence in a particular but illustrative case and we present a practical numerical algorithm which may be used to easily test and implement these ideas.

We are aware of two directly related numerical works. Baylis and Huschilt [6] used the same kind of idea to select, to a high precision, the initial condition corresponding to the physical solution, but the forward integration of the Lorentz–Dirac equation by a standard method was unable to avoid the appearance of runaway solutions as a consequence of numerical rounding-off errors. As we will see in the following, only a consistent application of the approximation method at each integration step is able to select the physical solution for large intervals of time. On the other hand, Bel [10] wrote a routine to construct the order reduction to which the solutions of the delay-differential equations (which describe the classical electrodynamics of two point charges when the radiation reaction is neglected)

tend. The aim here is to construct an equation of motion which describes the full electron motion (and not only its asymptotics) when there is no delay but the radiation reaction is taken into account.

In section 2 we review how the concept of order reduction and the hypothesis on the behaviour of the solutions when $\tau_0 \to 0$ lead to a Newtonian (i.e. second-order) equation of motion which has no runaway solution. In section 3 we gain a physical insight into the method of successive approximations and its convergence is analysed in two special cases. Section 4 describes the numerical algorithm used to implement the analytical method. To test the approximation method, in section 5 the routine is applied to the capture of a point charge by a Coulombian centre of opposite charge and in section 6 to the chaotic scattering of the same particle around three centres of the same charge. Some final remarks are collected in section 7.

2. The Newtonian equation of motion

Since the Lorentz–Dirac equation is obtained from conservation principles and most of its solutions are unphysical, it is reasonable to think that it is not the true equation of motion for the radiating classical electron, but only one among the conditions the physical solutions must fulfill. Several authors [2, 4] have shown that by imposing the asymptotic condition of null acceleration in the future infinity, the Lorentz–Dirac equation can be written in the form of an integro-differential equation which contains only physical solutions. But this is again a theory that may by applied only to the scattering process and for it to be useful one has to know the expression of the external force in terms of the proper time of the test charge, usually this dependence is unknown until the solution is found.

A more natural approach is provided by the concept of 'order reduction'. Order reductions have been used to replace the delay-differential equations which appears in the electrodynamics of two or more point charges [11, 12] and in nonlinear optics [13]. In the context of the Lorentz–Dirac equation this concept was partially contained in Landau and Lifshitz book [14] and was clearly discussed by Kerner [15] and Sanz [16]. The same concept has also been applied to analyse fourth-order equations that appear in theories of gravitation with a quadratic Lagrangian [17] and in the study of quantum corrections to Einstein equations [18].

The physical assumption is that the true equation of motion is a second-order equation

$$\ddot{x}^\mu = \xi^\mu(\tau, x, \dot{x}; \tau_0) \tag{4}$$

with the property that all its solutions satisfy the Lorentz–Dirac equation (1):

$$\xi^\mu = f^\mu + \tau_0 \left(\frac{\partial \xi^\mu}{\partial \tau} + \frac{\partial \xi^\mu}{\partial x^\nu}\dot{x}^\nu + \frac{\partial \xi^\mu}{\partial \dot{x}^\nu}\xi^\nu - \frac{\xi^\nu \xi_\nu}{c^2}\dot{x}^\mu \right). \tag{5}$$

In this way, selecting only the initial position and velocity will single out the corresponding physical solution. In fact, different numerical analyses [7, 8] support the hypothesis that Newtonian initial conditions are sufficient to obtain the physical solutions.

Of course, we have still to pick out the physical order reduction among the infinitely many equations (4) satisfying condition (5). To do it, we will use a remark that goes back at least to Bhabha [19]: the runaway solutions are singular in the limit $\tau_0 \to 0$ (or, equivalently, $e \to 0$). In consequence we will impose the condition

$$\lim_{\tau_0 \to 0} \xi^\mu = f^\mu \tag{6}$$

which will eliminate runaway solutions and select the solutions which satisfy the Newtonian equation $\ddot{x}^\mu = f^\mu$ when the radiation reaction is negligible. Of course, if f^μ is the Lorentz force per unit mass (2), we have to put it to equal 0 on the right-hand side of expression (6), but we want to keep the discussion a bit more general and consider also non-electromagnetic external forces.

Condition (5) is nonlinear and cannot be exactly solved except in rather trivial cases. In consequence an approximation scheme is necessary. The limit condition and the fact that τ_0 is very small in many situations of interest suggest using an expansion in powers of τ_0. This has been discussed in detail by Sanz [16] when f^μ is the Lorentz force. However, the convergence of the method has not been analysed and the complexity of the actual expressions for the expansion coefficients grows very fast. An alternative method is to construct a series of successive approximations $\ddot{x}^\mu = \xi_n^\mu$ (with $n = 0, 1, \ldots$) given by

$$\ddot{x}^\mu = \xi_0^\mu \equiv f^\mu \tag{7}$$

$$\ddot{x}^\mu = \xi_1^\mu \equiv f^\mu + \tau_0 \left(\frac{\partial f^\mu}{\partial \tau} + \frac{\partial f^\mu}{\partial x^\nu} \dot{x}^\nu + \frac{\partial f^\mu}{\partial \dot{x}^\nu} f^\nu - \frac{f^\nu f_\nu}{c^2} \dot{x}^\mu \right) \tag{8}$$

$$\ddot{x}^\mu = \xi_{n+1}^\mu \equiv f^\mu + \tau_0 \left(\frac{\partial \xi_n^\mu}{\partial \tau} + \frac{\partial \xi_n^\mu}{\partial x^\nu} \dot{x}^\nu + \frac{\partial \xi_n^\mu}{\partial \dot{x}^\nu} \xi_n^\nu - \frac{\xi_n^\nu \xi_{n\nu}}{c^2} \dot{x}^\mu \right). \tag{9}$$

By construction, the limit of this succession, if it exists, will satisfy conditions (5) and (6) and will be a Newtonian equation of motion with no runaway solutions. Of course, in practice, one could use one of the approximations, for a finite n, in the above sequence. Using equation (7) amounts to completely neglecting the radiation reaction and it is used in (8) to compute a first approximation to that reaction. Some textbooks [1, 14] suggest using the first approximation (8) and it has even been proposed as the exact equation of motion [20], but we do not want to impose *a priori* a limit to the approximation accuracy, which in practice may be estimated only if one is able to compute the next approximation to see if it is negligible or not.

Of course, the complexity of the explicit expressions in the sequence of approximations will grow even faster than in the power series method, but we are not interested here in the analytical expressions, instead we are interested in a numerical method to find the physical solutions of the Lorentz–Dirac equation, and the successive approximations are far more appropriate to construct the numerical algorithm than the power series.

Until now we have only considered the Lorentz–Dirac equation, but it should be obvious that nothing in the construction of successive approximations depends on the details of that equation. The same method may be applied, in principle, to any equation which has a small parameter and has a lower order when the latter vanishes [17, 18]. In particular, we are also interested in the non-relativistic limit of the Lorentz–Dirac equation and the Abraham-Lorentz equation:

$$\ddot{\boldsymbol{x}} = \boldsymbol{f} + \tau_0 \dddot{\boldsymbol{x}} \tag{10}$$

where a dot indicates derivative with respect to time t.

Before we discuss the numerical method, we will use this simpler equation to consider a couple of linear one-dimensional cases in which the sequence can be explicitly constructed and its convergence analysed. This will help us gain an insight into the method of successive approximations.

3. Two particular cases

3.1. Time-dependent force

Let us assume that the electron moves in a straight line under the action of a prescribed external force per unit mass $f(t)$. The Abraham–Lorentz equation

$$\ddot{x} = f(t) + \tau_0 \dddot{x} \tag{11}$$

can be equivalently written as follows

$$\ddot{x} = \int_0^\infty \mathrm{e}^{-u} f(t + \tau_0 u)\,\mathrm{d}u + C\mathrm{e}^{t/\tau_0} \tag{12}$$

where C is an arbitrary constant. This expression shows explicitly that most of the solutions will blow up as $t \to \infty$ and are singular in the limit $\tau_0 \to 0$. One may eliminate both problems by assuming that the true equation of motion is the one corresponding to the particular null value for the first integral C:

$$\ddot{x} = \int_0^\infty \mathrm{e}^{-u} f(t + \tau_0 u)\,\mathrm{d}u. \tag{13}$$

This is precisely the integro-differential equation used by several authors [2, 4]. On the other hand, the successive approximations of the previous section are

$$\ddot{x} = \xi_n(t) \equiv \sum_{k=0}^{n} \tau_0^k f^{(k)}(t) \tag{14}$$

and the exact order reduction is

$$\ddot{x} = \xi(t) \equiv \sum_{k=0}^{\infty} \tau_0^k f^{(k)}(t) \tag{15}$$

which is equivalent to (13), as can be seen by integrating term-by-term the Taylor expansion of $f(t + \tau_0 u)$.

3.2. Harmonic force

Let us now assume that the electron moves in a one-dimensional harmonic potential:

$$\ddot{x} = -\omega_0^2 x + \tau_0 \dddot{x}. \tag{16}$$

This linear equation may be written, in terms of the arbitrary constant C, as follows:

$$\ddot{x} + \gamma \dot{x} + \omega^2 x = C\mathrm{e}^{\lambda t} \tag{17}$$

where ω^2, γ and λ are the only real solutions of the equation set

$$\omega^2 = \omega_0^2 - \tau_0 \omega^2 \gamma \tag{18}$$

$$\gamma = \tau_0 \left(\omega^2 - \gamma^2\right) \tag{19}$$

$$\lambda = \frac{1}{\tau_0} + \gamma \tag{20}$$

and their explicit expressions (see the appendix) are given by

$$\gamma = \frac{(\alpha - 1)^2}{3\alpha\tau_0} \qquad \omega = \sqrt{\frac{\gamma}{\tau_0} + \gamma^2} \qquad \lambda = \frac{1}{\tau_0} + \gamma \tag{21}$$

with

$$\alpha \equiv \left[\sqrt{1+\tfrac{27}{4}\tau_0^2\omega_0^2}+\sqrt{\tfrac{27}{4}\tau_0^2\omega_0^2}\right]^{2/3}. \tag{22}$$

Since eliminating the runaway (singular) solutions amounts to taking $C = 0$ in equation (17), the true equation of motion will be

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0 \tag{23}$$

which, as expected, describes a damped harmonic oscillator whose frequency and damping coefficient differ from those corresponding to the free oscillator:

$$\omega = \omega_0[1 - \tfrac{1}{2}\tau_0^2\omega_0^2 + \mathrm{O}(\tau_0^4\omega_0^4)] \tag{24}$$

$$\gamma = \frac{1}{\tau_0}[\tau_0^2\omega_0^2 + \mathrm{O}(\tau_0^4\omega_0^4)]. \tag{25}$$

In this case, the successive approximations are all damped linear oscillators,

$$\ddot{x} + \gamma_n\dot{x} + \omega_n^2 x = 0 \qquad (n = 0, 1, \ldots) \tag{26}$$

and their coefficients satisfy the following recurrence:

$$\omega_{n+1}^2 = \omega_0^2 - \tau_0\omega_n^2\gamma_n \tag{27}$$

$$\gamma_{n+1} = \tau_0(\omega_n^2 - \gamma_n^2) \tag{28}$$

with $\gamma_0 = 0$. In consequence, if the sequence of approximations converge, the limit will be the oscillator (23) with ω and γ as defined by (18) and (19). We see, thus, that if the method is convergent it will produce the correct order reduction. Furthermore, in this particular case the method's convergence can be readily analysed because we can consider the discrete dynamical system given by the recurrence $(\omega_n^2, \gamma_n) \longrightarrow (\omega_{n+1}^2, \gamma_{n+1})$. As shown in the appendix, this recurrence has exactly one fixed point, the only real solution of (18) and (19), and the initial point $(\omega_0^2, 0)$ corresponds to an orbit that converges to that fixed point if and only if the following condition holds:

$$\tau_0\omega_0 < \sqrt{\frac{23+7\sqrt{13}}{54}} \simeq 0.95. \tag{29}$$

This proves that the successive approximations converge to the regular order reduction when (29) is satisfied.

In consequence, we see that the method of successive approximations gives the desired result for a large parameter range: it is enough to have a free oscillator period a few times higher than τ_0. Notice that the order reduction (23) exists for all values of ω_0, even if the corresponding fixed point is unstable. In the latter case the approximation method will fail to provide the desired order reduction, but this is not really surprising: even when a solution exists most approximation methods fail if some convergence conditions are not met.

Although this linear example is very simple, we hope that the result we have found (i.e. convergence of the method for a wide, though not full, parameter range) will remain true in other cases of interest. In the following section we will describe a numerical algorithm that allows us to check our hope and allows us to construct the approximations in other, more complex, cases.

4. The numerical iterative method

We have directly translated the iterative method (7)–(9) into a very general numerical code. The starting point is the embedded Runge–Kutta method of the order of 8 with automatic step-size control developed by Prince and Dormand in 1981 [21], as described by Hairer *et al* [22]. This is a continuous method which provides not only the values of the solution at discrete points but also a seventh-order polynomial which interpolates the solution between each pair of solution points.

In our algorithm each integration step is repeated iteratively. The first time an equation of the lower order is used, as in (7), and in each of the remaining iterations the interpolating polynomial that was computed in the previous one is used to compute the derivatives of the solution. This is the numerical equivalent of iteration (9). At each iteration an equation of the lower order is solved and the iterations are repeated either a prescribed number of times (this is equivalent to truncating the method for some given value of n) or until the difference between the estimations of the last two iterations is below some prescribed maximum relative error.

It is also possible to use a combination of the previous two as termination criterion and if the relative error requirement is not satisfied after the prescribed number of iterations have been performed, the routine halts with an error. In non-trivial circumstances this is a practical way to analyse the method's convergence. It also helps speed up the calculations with no loss of precision, because at each step the order of iteration (9) is only high enough to guarantee the desired accuracy. Of course, apart from this automatic selection of the iteration order, the routine also selects automatically the step size necessary to have the estimated truncation error below another prescribed tolerance.

We have checked this routine in a number of ways. First, we have written another two routines of a smaller order: a Dormand–Prince method of fifth order (which provides a fourth-order interpolating polynomial [23, 22]) and the classical Runge–Kutta method of fourth order. In the latter case the third-order interpolating polynomial is constructed by using Hermite interpolation. We have found that the three routines give the same results, but the required computing times vary greatly. In particular, the Runge–Kutta code is far less efficient and we have used it only to check the other two.

The accuracy of the numerical results have also been checked by using the known exact solution corresponding to the linear cases discussed in section 3. We have obtained a very good agreement, whose exact value depends on the prescribed maximum relative error.

Fortunately, it is always possible to check the results given by this method, even if the analytic solution is not known. After computing the desired solution forwards, it is enough to start from the endpoint and integrate the equation backwards. Since the iterative routine provides the endpoint, it is not necessary to start integrating backward from future infinity, or to estimate the final values which would correspond to the desired initial ones.

It should be stressed that the routines are very general and can be used with any equation or system of equations for which a method of successive approximations of this type is likely to converge.

5. A capture process

To show the method's performance we will first consider the classical capture of a relativistic point charge, e, by a static Coulombian centre of the opposite sign. In a system of units in which the unit length is the classical electron radius $r_0 \equiv e^2/mc^2$ and $c = 1$, if the initial position is $\boldsymbol{x}_0 = (50, 0)$ and the initial velocity is $\dot{\boldsymbol{x}}_0 = (0, 0.085)$, the orbits obtained with

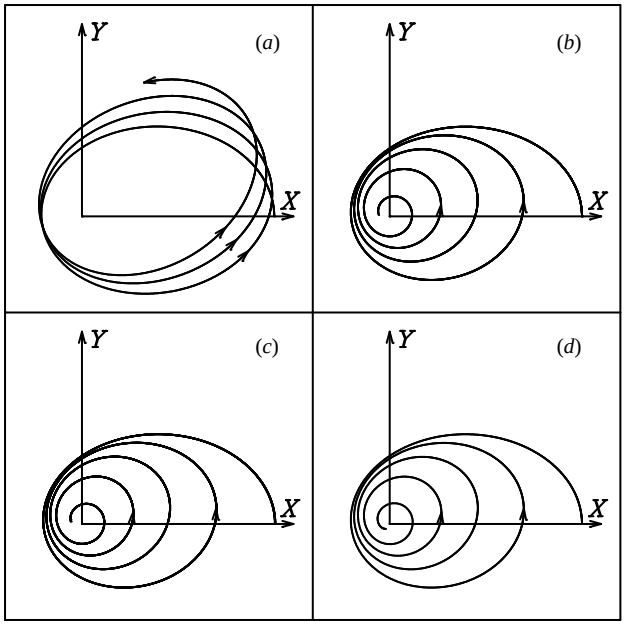

Figure 1. Capture of a point charge by a fixed charge of the opposite sign. (*a*) The radiation reaction is neglected. (*b*) The orbit is computed by integrating forward the Lorentz–Dirac equation. (*c*) The orbit is computed by integrating backward the Lorentz–Dirac equation. (*d*) The orbit is obtained by using the first iteration (8). The starting point is $(50r_0, 0)$.

different approximations are displayed in figure 1. When the radiation reaction is completely neglected (which amounts to using the Lorentz equations (2)–(7)) one obtains the special relativistic equivalent of the Keplerian ellipses: a precessing orbit.

In figure 1(*b*) there is an orbit that was computed by integrating forward the Lorentz–Dirac equation by allowing the routine to iterate the process until the relative difference between two consecutive approximations (7) is below 10^{-10}. The eccentricity loss which is predicted by Landau and Lifshitz [14] by making use of further approximations is apparent. The routine stops due to lack of convergence when the radial distance is below $3r_0$. This is not surprising and, as mentioned below, it only happens near the smallest distance which can be attained by a standard numerical routine before the numerical errors are too large.

Until now we have computed what we hope to be a good approximations to the capture orbit, we can start from its last point and integrate backward the exact Lorentz–Dirac equation with a standard routine to check if the previous solution satisfies that equation. The result is displayed in figure 1(*c*) and we see that both orbits are identical at the figure resolution.

Finally, by simply changing a couple of control parameters, we may use the same routine to construct numerically the first approximation (8), without having to compute and enter by hand the cumbersome explicit expression on its right-hand side. We see in figure 1(*d*) that it describes the capture rather well. The error is only higher than 1% in the last revolution. To get a lower relative error, more iterations are needed. In particular, for an error under 10^{-10} as described above, the routine had to iterate each step a number of times which varied from two (near the starting point) to more than 20 (in the final steps). We have depicted this solution to stress the fact that, although there is no convergence test in this case, the routine is not able to compute a much longer piece of orbit. Approximately after an additional quarter of a revolution it halts because of the big numerical instabilities induced by the high values of the acceleration near the Coulombian centre. At such small distances more sophisticated methods are necessary.

On the other hand, nothing guarantees that the order reduction (4) exists for all values of

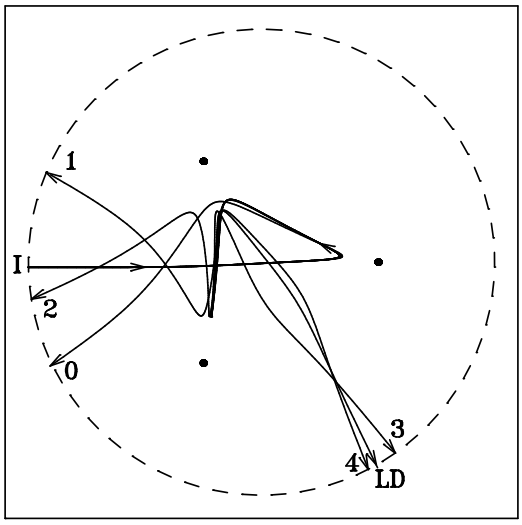

Figure 2. Scattering of a point charge by three equal fixed charges. The orbits are computed by using approximations (7)–(9) for $n = 0, 1, 2, 3, 4$ are labelled with the n value. Orbit LD is computed by integrating forward the Lorentz–Dirac equation and then checked by integrating backwards from the final point. The initial point for all forward orbits is labelled I.

the variables. For instance, if the capture process is one-dimensional, it has been shown [6] that all solutions are unphysical. Even if the orbit initially looks reasonable, it will finally invert its direction and become runaway. Thus, in this case we do not expect a global order reduction to exist and, in consequence, the numerical routine must fail at some point. In fact, if we take $(50, 0)$ as its initial velocity and integrate forward the Lorentz–Dirac equation, we find a solution which is very close to one corresponding to the Lorentz equation but the integration fails when the particle is below 4.5.

6. A scattering process

It is easy to find a case in which even a moderate error tolerance requires using high-order approximations. Let us consider the classical scattering of a point relativistic charge, e, by three static Coulombian centres of the same charge in the vertices of an equilateral triangle of side 10. The particle orbit starts from the point $\boldsymbol{x}_0 = (-20, -0.438\,75)$ with velocity $\dot{\boldsymbol{x}}_0 = (0.85, 0)$. Since a simple numerical experiment shows that the scattering is chaotic, we expect that any error in the numerical code or in the approximation method may introduce big changes in the computed orbit. Figure 2 shows that this is indeed the case, where the orbits computed by using approximations (7)–(9) for $n = 0, 1, 2, 3, 4$ are collected. We see that the convergence has yet to be reached. If we ask the routine to integrate forward the Lorentz–Dirac equation with high accuracy (relative difference between the last two approximations below 10^{-10}), the routine needs to use between 7 and 28 iterations at each step. The computed orbit is labelled LD and, once its final conditions have been computed, may be checked by integrating backward the exact Lorentz–Dirac equation: the agreement is very good and the result completely indistinguishable at the figure resolution.

Since the system has a sensitive dependence on the initial conditions, estimating the final configuration that produces, by backward integration, the selected initial conditions would require important computational work. For instance, the exit velocity when the particle crosses the circle of radius 20 again depicted in the figure is $|\dot{\boldsymbol{x}}| = 0.69$, which is very different from the initial value.

Since we are able to integrate forward the Lorentz–Dirac equation, it is now easy to compute scattering functions. Let us assume that the test particle is sent from the points

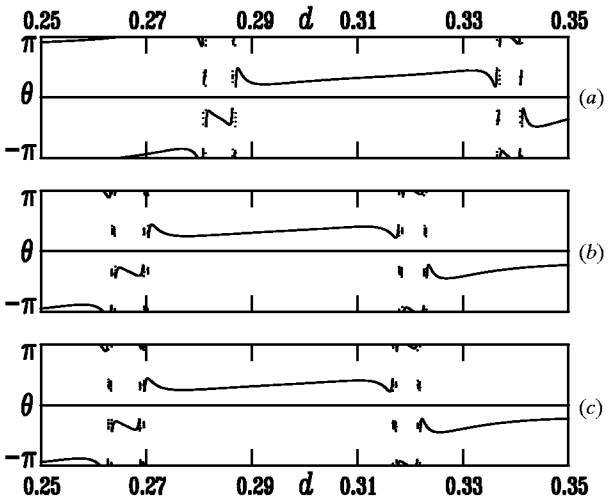

Figure 3. Scattering function $\theta(d)$ when (a) the radiation reaction is neglected, (b) the first iteration (8) is used, and (c) the Lorentz–Dirac equation is solved forwards.

$\boldsymbol{x}_0 = (-50, d)$ with velocity $\dot{\boldsymbol{x}}_0 = (0.85, 0)$ for 10 000 values of the impact parameter $0.25 \leqslant d \leqslant 0.35$. We will compute the angle, θ, between the velocity and the x-axis when the particle exits the scattering region (which we define somewhat arbitrarily by the condition $|\boldsymbol{x}| < 50$). In figure 3(a) the scattering angle $\theta(d)$ is displayed, we obtain it when the radiation reaction is neglected and the Lorentz equation (7) is used. The characteristic discontinuities of chaotic scattering are clearly visible and may be confirmed if successive blow-ups are computed.

Chaotic scattering is usually studied in conservative systems and its origin is tied to the existence of a chaotic non-attracting set of unstable periodic orbits that remain in the scattering region forever [9]. Solutions that get close to one of these periodic orbits will remain near it for a long time and the accumulated defocusing effect of successive reflections around scattering centres will produce a sensitive dependence on the initial conditions. It would be interesting to see what happens when the radiation reaction is taken into account. One expects that the energy loss will destroy most of the periodic orbits but, on the other hand, the possibility of remaining in the scattering region for long times still exists. In figure 3(b) we display the scattering function that is obtained when the first approximation (8) is used and in figure 3(c) we display the result corresponding to the forward integration of the Lorentz–Dirac equation with the same high accuracy mentioned above. We see that from a qualitative point of view the radiation reaction appears as a shift of the scattering function $\theta(d)$ and that the first iteration (8) slightly underestimates the exact value of this shift. A more detailed study of the chaotic scattering is beyond the scope of this work and will be described elsewhere.

7. Conclusions

It has been shown that equation hierarchies of the form (7)–(9) are not only of theoretical interest but may be used to solve numerical problems in practice. A linear example in which everything may be analytically computed has been analysed to gain some insight into the general problem. We have described a set of routines which may be used to test with great ease the convergence of the method and to compute with low effort some solutions of the

Lorentz–Dirac equation (and equations with the same structure [17, 18]) that are difficult or impossible to obtain by means of previously described methods.

As expected from the analysis in section 3, we have found that the routine does not always converge, but it performs surprisingly well in many cases and no runaway solution is induced by numerical errors.

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Appendix

We will study here the discrete dynamical system $(\omega_n^2, \gamma_n) \longrightarrow (\omega_{n+1}^2, \gamma_{n+1})$ defined in (27), (28). To simplify the notation we will introduce the non-dimensional quantities $x \equiv \tau_0^2\omega^2$ and $y \equiv \tau_0\gamma$ as dynamical variables. The recurrence is then

$$x_{n+1} = x_0 - x_n y_n \tag{30}$$

$$y_{n+1} = x_n - y_n^2 \tag{31}$$

with $x_0 = \tau_0^2\omega_0^2$ and $y_0 = 0$. The fixed points of this dynamical systems aregiven by

$$x = x_0 - xy \tag{32}$$

$$y = x - y^2 \tag{33}$$

or, equivalently, by

$$y(1+y)^2 = x_0 \tag{34}$$

$$x = y(1+y). \tag{35}$$

For $x_0 > 0$ the cubic (34) has a single real root, which happens to be positive and given by

$$y = \frac{(\alpha-1)^2}{3\alpha} \qquad \alpha \equiv \left[\sqrt{1+\tfrac{27}{4}x_0} + \sqrt{\tfrac{27}{4}x_0}\right]^{2/3}. \tag{36}$$

In consequence, the dynamical system has exactly one fixed point. To study its stability we have to compute its characteristic multipliers, i.e. the eigenvalues of the differential of the map $(x, y) \longrightarrow (x_0 - xy, x - y^2)$, which can be written as follows:

$$\lambda = \frac{-3y \pm \sqrt{-y(4+3y)}}{2}. \tag{37}$$

Since y is given in equation (36), these eigenvalues are complex and the condition for asymptotic stability, $|\lambda| < 1$, is $y < (\sqrt{13}-1)/6$ or, as a consequence of (34), $x_0 < (23+7\sqrt{13})/54$. The latter is precisely condition (29). We have checked numerically that $(x_0, 0)$ is in the basin of attraction of the fixed point and the corresponding orbit does in fact tend to the fixed point when the former condition holds.

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